5.2: The Characteristic equation

Key idea: For any square matrix A, $\det(A-\lambda I)$ is a polynomial the roots of which are the eigenvalues of A.

Today's goal is to calculate the eigenvalues of a given matrix. To do this note λ is an eigenvalue of A if:

$A\vec{x} = \lambda\vec{x}$ has a nontrivial sol'n $\Rightarrow$ $(A-\lambda I)\vec{x} = \vec{0}$ has a nontrivial solution $\Rightarrow$ $A-\lambda I$ is not invertible $\Rightarrow$ $\underbrace{\det(A-\lambda I) = 0}_{\text{the characteristic equation of } A}$

Now, $\det(A-\lambda I)$ is a polynomial, and we see it's zeros are the eigenvalues of A. Before defining this formally we consider an example.

Ex Find all eigenvalues of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. (Recall $\lambda_1 = 7, \lambda_2 = -4$ from 5.1)

From above λ is an eigenvalue if $\det(A-\lambda I) = 0$. So we find $A-\lambda I$ and $\det(A-\lambda I)$:

$$A - \lambda I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{bmatrix}$$

$$\Rightarrow \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - (6)(5) = 2 - 3\lambda + \lambda^2 - 30$$
$$= \underbrace{\lambda^2 - 3\lambda - 28}_{\text{characteristic polynomial of } A}$$

So $\underbrace{\det(A-\lambda I) = 0}_{\text{(characteristic eq'n)}} \iff \underbrace{\lambda^2 - 3\lambda - 28 = 0}_{\text{(characteristic eq'n)}}$

$\Rightarrow \lambda$ is an eigenvalue of A if $(\lambda-7)(\lambda+4) = 0$

$\Rightarrow \lambda_1 = 7, \lambda_2 = -4$ are the eigenvalues of A.

Before another example, we establish definitions:

λ is an eigenvalue if and only if:

- λ satisfies the characteristic equation of A: $\det(A-\lambda I)=0$.
- λ is a root of the characteristic polynomial of A: $\det(A-\lambda I)$.

The multiplicity of λ is its algebraic multiplicity as a root of the char. poly.

Ex| Find and state the multiplicities of all eigenvalues of

$$A=\begin{bmatrix} 5 & 3 & 0 & 0 & 7 & 3 \\ 0 & 3 & 1 & 2 & 5 & 1 \\ 0 & 0 & 5 & 8 & 5 & 6 \\ 0 & 0 & 0 & -1 & -9 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \rightsquigarrow A-\lambda I = \begin{bmatrix} 5-\lambda & 3 & 0 & 0 & 7 & 3 \\ 0 & 3-\lambda & 1 & 2 & 5 & 1 \\ 0 & 0 & 5-\lambda & 8 & 5 & 6 \\ 0 & 0 & 0 & -1-\lambda & -9 & 0 \\ 0 & 0 & 0 & 0 & -1-\lambda & 2 \\ 0 & 0 & 0 & 0 & 0 & 5-\lambda \end{bmatrix}$$

$$\Rightarrow \det(A-\lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(-1-\lambda)(-1-\lambda)(5-\lambda)$$
$$= (3-\lambda)(-1-\lambda)^2(5-\lambda)^3$$

(because A, $A-\lambda I$ are triangular)

Clearly the roots of the characteristic polynomial of A are

$$\begin{array}{lccc} & \lambda_1 = 3 & \lambda_2 = -1 & \lambda_3 = 5 \\ & \downarrow & \downarrow & \downarrow \\ \text{mult.:} & 1 & 2 & 3 \end{array}$$

(-1 is a double root
5 is a triple root)

Observe from the previous example we see:

Fact: If A is a triangular matrix then the eigenvalues of A are the diagonal entries repeated to respect multiplicity.

To finish we consider a 3×3 matrix with only one real eigenvalue.

Ex] Consider $A = \begin{bmatrix} 1 & 0 & -2 \\ 4 & 3 & 5 \\ 2 & 0 & 1 \end{bmatrix}$. $\quad A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -2 \\ 4 & 3-\lambda & 5 \\ 2 & 0 & 1-\lambda \end{bmatrix}$

$$\Rightarrow \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 4 & 3-\lambda & 5 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left((1-\lambda)^2 + 4\right)$$

$$= (3-\lambda)(\lambda^2 - 2\lambda + 5)$$

So the real eigenvalues of A are λ s.t.

$$(3-\lambda)(\lambda^2 - 2\lambda + 5) = 0$$

$$(3-\lambda) = 0 \Rightarrow \lambda_1 = 2$$

$$(\lambda^2 - 2\lambda + 5) = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \tfrac{1}{2}\sqrt{-16} = 1 \pm 2i.$$

So A has one real eigenvalue $\lambda_1 = 2$ not real.

and two complex eigenvalues $\lambda_2 = 1 + 2i$, $\lambda_3 = 1 - 2i$.

In our class we will be minimally concerned with complex eigenvalues.